FIGURING IT OUT: MATHEMATICAL LEARNING AS GUIDED SEMIOTIC DISAMBIGUATION OF USEFUL YET INITIALLY ENTANGLED INTUITIONS

Dor Abrahamson
University of California, Berkeley
dor@berkeley.edu

Michael Bryant
University of California, Berkeley
mbryant@berkeley.edu

José F. Gutierrez
University of California, Berkeley
josefrancisco@berkeley.edu

Amit V. Mookerjee
University of California, Berkeley
mookerjee@berkeley.edu

Daria Souchkova
University of California, Berkeley
g0wings19@yahoo.com

Ian Thacker
University of California, Berkeley
thackeri@berkeley.edu

When participants in inquiry refer to an object, they may, unbeknown to them, construct the object differently. They thus tacitly attribute different idiosyncratic senses for their respective constructions and consequently draw different inferences regarding the phenomenon under investigation. A single person, too, may shift between alternative constructions of a mathematical object, assigning them different senses, thus arriving at apparently competing conclusions. Only upon acknowledging the different constructions can the person begin to explore whether and how the differing conclusions are in fact complementary. Building on empirical data of students engaged in interview-based tutorial activities targeting fundamental probability notions, we explicate breakdowns such false-contradiction introduces into learning processes yet suggest opportunities such ambiguity fosters.

'Seeing as....' is not part of perception. And for that reason it is like seeing and again not like. (Wittgenstein, 1953)

Objectives

Objects per se do not carry any meaning—all meaning is mentally constructed. The same principle holds for classroom learning materials, be these plastic tokens, spatial–numerical diagrams, or symbolic inscriptions. Yet this fundamental tenet of phenomenology and constructivism—that meanings of objects are mediated by implicit mental structures and are anyhow transparent in the ongoing *Dasein* of goal-oriented activity—may be difficult for a teacher to bear in mind let alone apply successfully in the real-time contingencies of engaged mathematics discourse. Moreover, students are often unaware of the constructed nature of their own mathematical perception of objects and therefore do not differentiate between objects per se (the distal stimuli) and their personal constructions of these objects (the proximal stimuli) (Wittgenstein, 1953). Consequently, teachers and students may be explicitly speaking about the same object yet implicitly ascribing to it diverging meanings and related inferential implications, and therefore their communication fidelity is a priori compromised (e.g., Borovcnik & Bentz, 1991). Nevertheless, a teacher can be well aware that two or more students are seeing a mathematical object differently even though they are using similar lexical labels to index the same object, and a skilled teacher can capitalize on these covert ambiguities to orchestrate productive discursive negotiations (Moschkovich, 2008). Still, teachers cannot always interpret, monitor, foster, or amend students' idiosyncratic constructions so that they accord sufficiently with normative constructions (i.e., so that the meanings are taken-as-shared, Cobb, 2005). Thus, covert communication breakdowns in classroom discourse may be more ubiquitous than one might expect, with interlocutors bearing personal meanings that overlap just enough to preclude overt breakdown.

Yet is such covert polysemy and the communication breakdowns it engenders necessarily detrimental to learning? Here we wish to argue that some covert semiotic "fuzziness" may in fact ultimately support collaborative learning, because it enables interlocutors the ostensible intersubjectivity requisite of mutually supportive discourse, *even as they are seeing objects differently*. Specifically, when students construct differently a semiotic artifact under joint inquiry, they may contribute to a conversation different mathematically valid assertions precisely because they are not cognizant of their different constructions. For example, if you and I are gazing at an array of six dots, I may see it as two rows of three dots each even as you see it as three columns of two dots each. Referring to the array, I might say that, "It is two times three," but then you might disagree that, "It is three times two." Notably the "it" in each of our respective utterances does not refer to the "objective" array itself but to our respective mental constructions of the array. Sorting out disagreement over the meaning of objects thus becomes an opportunity to co-examine the semiotic elements implicit to the conversation, e.g., the unitized groups of dots in the array. Namely, the conversation may shift from arguing over some ill-defined *mis*taken-as-shared 'it' to speaking about how we are seeing 'it,' i.e., to *figuring it out.* So doing, we may discover the semiotic contingencies of our respective statements and formulate a mathematical assertion that reconciles their respective meanings in the form of the targeted mathematical content of the instructional activity, e.g., we may discover that "2 x 3 = 3 x 2."

This paper examines excerpts from one-to-one interview-based conversations between a researcher and three students, in which the students each sustained throughout a tutorial activity two different framings of a single iconic artifact, which they had been guided to construct so as to model a mathematical system under inquiry. Each framing of the object implied a different expectation for the behavior of this system, and the students' expectations shifted with their framing of the object. We argue that both mental constructions of the object were mathematically correct, if naively worded, and that the students were able to reconcile these constructions successfully only if they were aware of the contingency of their assertions on their implicit framings of the object. We further submit that the ambiguity of the object ultimately supported these students' learning, because it elicited the two key idea elements of the targeted mathematical notion and juxtaposed them for reflection. That is, embedding key idea elements of a targeted mathematical notion within a single semiotic artifact instantiated these elements as co-present in the problem space, thus honing a generative confusion that supported the conjoining of these idea elements into the targeted conceptual composite. Thus we support *embracing diverse perspectives*, in line with the conference theme.

Theoretical Framework

In his *Philosophical Investigations,* Wittgenstein (1953) sets out by describing a Tower-of-Babel scene, in which construction workers are able to collaborate only because they share referents for their otherwise arbitrary verbal utterances. Thus, if I ask for a "brick" and you hand me a brick, we are capable of co-constructing an artifact, but if you instead handed me a bucket, the premise of our collaboration would be compromised. Yet along with my frustration resulting from this patent miscommunication, we maintain, I may gain a useful realization of the language game underlying human intersubjectivity. Namely, as language breaks down, its normatively obscure equipmentality is disclosed for scrutiny (Heidegger, 1962). To the extent that language, writ large, is the internalized vehicle of human reasoning (Vygotsky, 1934/1962), understanding its semiotic mediation of "objective" situations may be instrumental to reflecting on one's learning process, which necessarily requires the adoption of cultural forms of seeing and referring to aspects of one's personal, unreified phenomenology (Bamberger & diSessa, 2003; Goodwin, 1994; Stevens & Hall, 1998).

In the brick-vs.-bucket communication breakdown, above, one and the same verbal utterance, "brick," differentially referred to two objects in the joint perceptual field. Yet inherent to this miscommunication is that one and the same object was interpreted differentially—the object that you saw as a brick, I saw as a bucket. Such flagrantly conflicted constructions of distal stimuli, though reserved for rhetorical effect in philosophical discourse, may nevertheless underlie—if in a more nuanced caliber—challenges inherent to instructional discourse. In the case of the disciplines where unequivocal definitions are paramount to the production of texts (in the continental, multi-modal sense of 'text'), it thus becomes important to monitor for shared meanings of objects.

Note that the sense that interlocutors ascribe to objects are not necessarily personally available-it is not the case that students are consistently conscious of *how* they are seeing an object, even as they are capable of describing what the object *means* in the context of disciplinary discourse, such as problem solving. Indeed, idiosyncratic constructions of objects may be by-and-large inaccessible ('cognitively impenetrable,' Pylyshyn, 1973), unlike meanings, which may be verbally couched as rationalized inferences pertaining to a phenomenon under inquiry. Nevertheless, the very rationale of scholarly inquiry into students' understanding of instructional materials is the identification and articulation of their personal constructions of objects. This problematique of an analytic endeavor to name the ineffable psychological facets of human discourse has been treated before:

> We do not claim to make clear and explicit what the users of the unclear expression had unconsciously in mind all along. We do not expose hidden meanings, as the words 'analysis' and 'explication' would suggest; we supply lacks. We fix on the particular functions of the unclear expression that make it worthy troubling about, and then devise a substitute, clear and couched in terms of our liking, that fills those functions. (Quine, 1960, pp. 258-259)

Whereas ambiguity of discourse readily suggests *inter*subjective situations, Quine orients us toward *intra*subjective situations. Namely, by virtue of referring to an object by two different labels, one perforce brings out different meanings, demonstrating a phenomenon Quine called *intrasubjective stimulus synonymy*. For example, "For each speaker, 'Bachelor' and 'Unmarried man' are stimulus-synonymous without having the same meaning in any acceptable defined sense of 'meaning'" (Quine, 1960, p. 46).

In this paper, however, we present cases of intrasubjective stimulus *polysemy* and discuss their consequences for mathematical learning. Namely, we demonstrate how an individual student's competing perceptual constructions of a mathematical semiotic artifact initially create cognitive conflict between two inferences that are in fact both mathematically correct. These inferences appear to the student as conflicting, rather than complementary, because the student tacitly equates the objective artifact with its perceptual construction. We highlight the indispensable role of instructional designers and mathematics teachers in both eliciting from students each of the apparently conflicting inferences and facilitating discourse that aims at exposing the different perceptual constructions underlying each inference. The vocabulary, constructs, and definitions necessarily generated so as to achieve these disambiguations are pivotal aims of the instructional process, because these discursive tools help students *synthesize* (Schön, 1981) tacit and mathematical views of the instructional materials. Yet what are the implications of this thesis for teachers' practice?

Guiding students to construe mathematical objects in accord with disciplinary norms is generally an asymmetric process, in which a teacher enables a student to see things as she does and any alternative construction falls by the wayside (Goodwin, 1994; Stevens & Hall, 1998). Yet for some disciplinary content topics, multiple views of problems are intrinsic to mathematical discourse, so that fostering such ambiguity in classroom discourse may play a nurturing, rather than an obstructing role. For example, a sequence of coin tosses—Heads, Tails, Heads, Tails (HTHT)—may be construed as one of sixteen equiprobable elemental events in the sample space of the four-coin-flips experiment (1/16) or, alternatively, as the aggregate event "2 Heads and 2 Tails in any order" that has a 6/16 chance of occurring (on the contingency of mathematical definitions on social contract, see Barnes, Henry, & Bloor, 1996; Ernest, 2008; Weisstein, 2006). A student may sense that one must decide between these two mathematically valid constructions, thinking that HTHT cannot have both 1/16 and 6/16 chances of occurring. Namely, this student would experience a need to decide whether the object—the distal stimulus presented by the inscription "HTHT"—has this value or that value for the property of likelihood, where in fact the student is implicitly referring to different percepts but not articulating the implications of attending or not attending to the internal order of the four singleton events (Abrahamson, 2009).

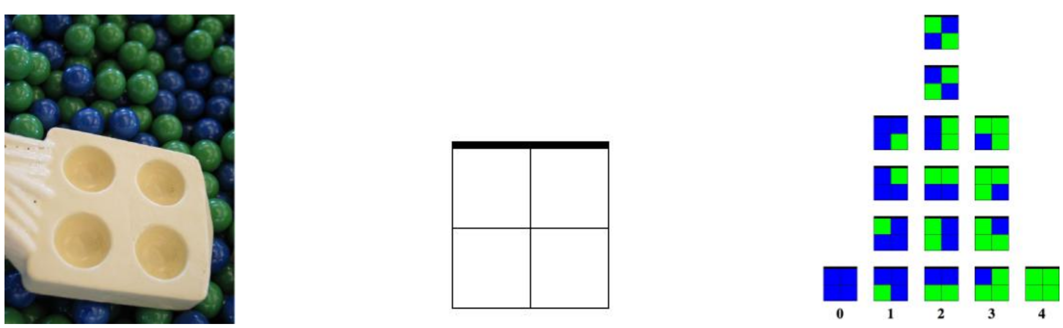
Indeed, in this paper, we present empirical data to argue that one challenge inherent to supporting students' sense making processes is that students are liable to implicitly equate mental constructions with objects per se and thus experience difficulty accepting, let alone reconciling, any competing meanings they may attribute to these objects. That is, when the students think they must make up their mind with respect to the assertions they express about a mathematical object, in fact these different assertions are not necessarily mutually exclusive but possibly complementary, because each assertion refers to a different mental construction of one and the same object. Differentiating these assertions on the basis of their underlying perpetual constructions is crucial for conceptual development in those cases where both assertions are conceptually pertinent. For example, acknowledging the ambiguity of HTHT may help a student understand that the probability of an aggregate event is the sum total of the probabilities of its elemental events (1/16 + 1/16 + 1/16 + 1/16 + 1/16 + 1/16 = 6/16).

Background, Methods, and Research Focus

The episodes analyzed herein come from a larger corpus of data collected over a succession of cumulative studies conducted as part of the *Seeing Chance* project to understand and promote probability learning (Embodied Design Research Laboratory, UC Berkeley). Specifically, we examine the behavior of 3 out of 28 middle-school participants in Abrahamson (2009). The study took place in a private school in the SF East Bay area (33% on financial aid; 10% minority students), and all three focus students for this paper were ranked by their mathematics teachers as high achieving. The phenomenon of intrasubjective stimulus polysemy that we examine here was typical of all students, yet it elicited longer, richer, and more articulated deliberations from the older and higher-achieving students—perhaps because these students were more self-monitoring and self-exacting in their mathematical reasoning—and hence these study participants help us understand what may be a ubiquitous phenomenon characteristic of all students. Each student participated in a semi-structured clinical interview that lasted about one hour.

The project was conducted in the design-based research approach, which typically examines some conjecture as to an underlying mechanism inherent to a hypothetical learning phenomenon by creating empirical contexts in which to examine this conjecture (Confrey, 2005). Emerging from study cycles of design, empirical implementation, and analysis, in which the researchers tune the learning environment and, reciprocally, their emerging understandings of learning phenomena, are new instructional materials or principles as well as ‘ontological innovations’ (diSessa & Cobb, 2004), theoretical constructs that capture consistent patterns that the researchers discover in the empirical data. This paper is about intrasubjective stimulus polysemy, an ontological innovation that we are proposing.

Central to the interview was a set of instructional materials designed to elicit students’ population-to-sample informal inferences, which are mathematically correct though only qualitative and unwarranted by mathematical argumentation. Students are then guided to construct the expanded sample space of this experiment as a means of creating a context for the dyad to discuss differences in how natural perceptual inclination and formal mathematical analysis couch inferences with regard to probabilistic behavior of random generators. Here we will introduce only those materials that feature in the data under inquiry. The interview begins by showing participants a tub containing many green and blue marbles of equal numbers as well as a *marbles scooper* (see Figure 1a), a utensil for drawing out of the box a sample with a precise number of marbles that are spatially arranged in a particular permutation. Strictly speaking, this is a hypergeometric (without replacement) problem, yet the large population-to-sample ratio enables us to treat it as an approximation for the binomial. Participants are asked to offer their guess for the distribution of outcomes in a hypothetical experiment with this random generator. Next, participants are given a set of blank cards with a 2-by-2 table structurally resembling the scooper (see Figure 1b) and are guided to construct the sample space of the experiment and assemble in the form of the *combinations tower* (see Figure 1c).



a. *b.* *c.*

Figure 1. (a) The marble scooper; (b) one of many cards for conducting combinatorial analysis of

Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.

the experiment; and (c) the combinations tower—an assembly of the sample space in a format designed to resonate with students' inferences for the experiment.

Whereas students by-and-large guessed correctly that the plurality of experimental outcomes would be of type "2 green and 2 blue [in any order]" (hence 2g2b), they experienced difficulty in appreciating why analytic attention to the *order* of the four singleton events in each scoop may be advantageous to supporting their guess. Nevertheless, once they had completed constructing the combinations tower, participants appropriated this structure as a warrant for their guess by indexing the relatively greater number of 2g2b elemental events as compared to other aggregate events. In previous publications we attributed students' reluctance to attend to the combinations' internal order to a tension between tacit and mathematical constructions of the sample: whereas students naturally couch the experiment in terms of five (aggregate) events (no green, 1 green, 2 green, 3 green, and 4 green), combinatorial analysis requires attention also to the internal order of the four singleton events and therefore produces sixteen (elemental) events.

The current study focused on interview episodes in which participants switch between aggregate- and elemental-event constructions of a compound-event card containing four singleton events. We compared these episodes in an attempt to explore for relations between the participants' awareness of their constructions and their success in coordinating the tacit and mathematical formulations of the anticipated experimental outcome distribution.

Results and Analyses

Table 1, below, offers a preview of our results. For rhetorical clarity, we use the familiar duck-rabbit ambiguous figure. (Joseph Jastrow popularized it in the late 19th century so as to illustrate perceptual agency in constructing distal stimuli.) If a viewer is asked to infer the eating habits of this ambiguous creature, yet the viewer is unaware that his mental construction of the image keeps shifting ("duck…no, rabbit!"), then the viewer will not understand his vacillating inferences ("fish…no, carrots!") and will take this inconsistency as marking confusion. If, however, the viewer can label each mental construction of this object as well as their critical disambiguating features ("beak…ears"), then the viewer will be equanimous with respect to his conflicting inferences (cf. Tsal & Kolbet, 1985).

Table 1. *Inferential Reasoning for an Overtly Ambiguous Figure*

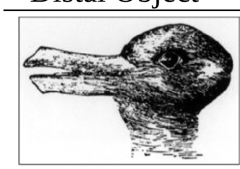
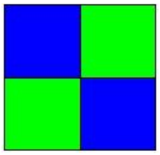
Distal Object	Disambiguating Features	Proximal Object	Inference for Diet
	Beak	Duck	Fish
	Ears	Rabbit	Carrots

Table 2, below, presents the less familiar case, from probability studies, of a compound event as an ambiguous figure. A viewer who attends to the particular configuration of green and blue cells in this object may construct it as one of sixteen unique equiprobable elemental events in the sample space. However, a viewer who ignores the internal order of cells in this object and constructs it as 2g2b may interpret it as the aggregate event most likely to occur in the marbles-scooping experiment. If, however, the viewer is unaware of her shifting personal constructions, she will interpret her shifting inferences as marking confusion.

Table 2. *Inferential Reasoning for a Covertly Ambiguous Figure*

Distal Object	Disambiguating Features	Proximal Object	Inference for Distribution
	Order	Elemental event	Equiprobable
	Number	Aggregate event	Heteroprobable

Each of the following three 6th-grade students, Lavi, Sima, and Razi, identified the completed combinations tower as resonating with their mathematically correct guesses for the outcome distribution of the marbles-scooping experiment. However, subsequent discussion suggested that their insight was unstably based on a global perception of relations among the combinations-tower columns and that they were still struggling to align their insight with the ambiguous construction of the combinations tower's constituent elements.

Lavi: "My Mind's Going Back and Forth"

The interviewer lifts out of the sample space two cards—one of the 3g1b cards and the 4g card—and asks Lavi to compare their likelihoods. The following conversation ensues:

Lavi: There's only four [ways] for getting three [3g1b]. I guess it wouldn't be chance…Oh, I guess it would be chance. And then there's only one way that there can be four [4g].

Res.: So, what do you mean [by] "It's not chance" and "It is chance?"

Lavi: Ah, I don't know, that thought just kind of [popped] into my mind and I just let it come out.

When Lavi says, "It wouldn't be chance," he is viewing the individual cards as representing heteroprobable aggregate events whose chance is indexed by the number of permutations in their respective columns. When he says, "It would be chance," he is viewing these same cards as equiprobable elemental events for which only chance, not logic, would cause greater frequency. The compound event is thus a physical object imbued with different mathematical constructions, and Lavi alternately refers to these competing constructions. However, he does not appear to realize that he is shifting his point of view, so he is confused.

The interviewer repeats the question for another pair of cards. Lavi asks whether he should take these cards to mean "a specific card or an amount of each color" and claims that all "specific cards" are equally likely. The interviewer asks Lavi to compare the cards on the basis of "the amount," i.e., to ignore placement. After some hesitation, Lavi nevertheless asserts, "It is chance," invoking the randomness of the sampling device (on the 'equiprobability bias,' see Falk & Lann, 2008; LeCoutre, 1992). Throughout a subsequent series of questions, Lavi vacillates between viewing individual cards as "ducks" or "rabbits," coming just short of reconciliation.

Sima: Stuck on Rabbit

Like Lavi, Sima begins by articulating the equiprobability of the sixteen compound events. She creates the term "color-wise" to refer to the groups and "place-wise" to refer to individual cards and states that “color-wise” the groups have different probabilities but place-wise “they’re equivalent.” Yet, once the interviewer asks her to compare two cards selected from different columns, she maintains that they have different likelihoods. Subsequently, she appears to experience difficulty in dislodging from the aggregate view and returning to the elemental view—she is “stuck on rabbit” and insists that any 2g2b card is more likely than any 3g1b card. Only after the interviewer simulates random sampling from these sixteen cards and refers back to the initial experiment is Sima able to reassume equiprobability.

Razi: Chooses Rabbit

Like Lavi and Sima, Razy articulates that specific cards are equally likely yet that viewing them by the “number of each color” makes some cards more likely than others. She appears to command greater fluency than Lavi and Sima in shifting between the competing constructions of the events, yet she incurs greater difficulty in articulating the implications of each view for the outcome distribution.

Razi: The majority of the scoops would come out with two blues and two greens.

Res.: A moment ago you told me that each pattern has the same likelihood to show up. Is their a contradiction here?

Razi: Yes and no. Before I said “each specific pattern.” Now I’m saying each pattern with two blues and two greens….

Res.: But do you still hold to the fact that each exact pattern has the same chance?

Razi: I am not sure.

Finally, when asked to compare another two cards, Razi becomes entrenched in the aggregate view. It appears that whereas Razi understands that there are two ways to see the object, she feels she must choose between “duck” and “rabbit.”

Conclusion

Students’ awareness of their perceptual constructions of ambiguous mathematical objects—their intrasubjective stimulus polysemy—impacts their capacity to generate domain-specific constructs and, in turn, to coordinate tacit and analytic formulations of situated phenomena toward deep conceptual understanding. We have demonstrated this relation for the case of the binomial and will continue to pursue our conjecture as it plays out in the learning of other mathematical concepts.

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